ME 4555 – Lecture 11 – The Laplace Transform

The Laplace transform is a <u>tool</u> for solving linear ODEs, i.e. the ODEs that correspond to LTI systems.
(linear time-invariant)

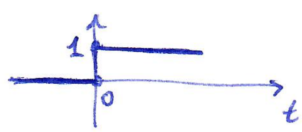
The Laplace transform takes a function of time $(t \geq 0)$, say $f(t)$, and produces a function of "s" ($s \in \mathbb{C}$, a complex number). The definition is:

$$\boxed{F(s) \equiv \mathcal{L}\{f(t)\} = \underbrace{\int_{0^-}^{\infty} f(t)\, e^{-st}\, dt}}$$

[Laplace transforms are often denoted using upper-case letters]

{technically, it's $\lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\infty} (\dots)$ to ensure $t=0$ is included}

<u>Ex</u>: let $f(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$



also called the Heaviside function (unit step) and often denoted $H(t)$.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt = \int_0^{\infty} e^{-st}\, dt = \left[-\tfrac{1}{s} e^{-st}\right]_0^{\infty} = \frac{1}{s}$$

as long as real part of s is positive ($\mathrm{Re}(s) > 0$)

Therefore, $\boxed{\mathcal{L}\{H(t)\} = \frac{1}{s}}$

The Laplace transform is <u>linear</u>.

<u>Homogeneity</u>: $\mathcal{L}\{a \cdot f(t)\} = \int_0^\infty a f(t) e^{-st} dt = a \int_0^\infty f(t) e^{-st} dt = a \mathcal{L}\{f(t)\}$

<u>Superposition</u> $\mathcal{L}\{f(t) + g(t)\} = \int_0^\infty (f(t) + g(t)) e^{-st} dt$

$$= \int_0^\infty f(t) e^{-st} dt + \int_0^\infty g(t) e^{-st} dt$$

$$= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}.$$

Laplace transform of a derivative:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st} dt$$

(integration by parts.
$u = f \quad v = e^{-st}$
$u' = f' \quad v' = -se^{-st}$)

$$= \left[f(t) e^{-st}\right]_0^\infty - \int_0^\infty f(t) (-s e^{-st}) dt.$$

$$= -f(0) + s \int_0^\infty f(t) e^{-st} dt$$

$$= s \mathcal{L}\{f(t)\} - f(0).$$

So if $\mathcal{L}\{f(t)\} = F(s)$

then $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

Table of a few Laplace transforms

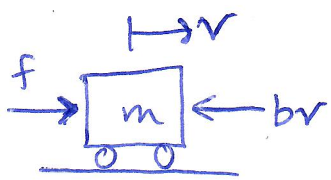
$f(t)$	$\xrightarrow{\mathcal{L}\{\cdot\}}$ $\xleftarrow[\mathcal{L}^{-1}\{\cdot\}]{}$	$F(s)$
$H(t)$ (unit step)		$\frac{1}{s}$
$a\,f(t)$		$aF(s)$
$f(t)+g(t)$		$F(s)+G(s)$
$\frac{df(t)}{dt}$		$sF(s)-f(0)$
$\frac{d^2f(t)}{dt^2}$		$s^2F(s)-sf(0)-f'(0)$
e^{-at}		$\frac{1}{s+a}$
$\frac{1}{a}(1-e^{-at})$		$\frac{1}{s(s+a)}$

(Linearity)

Note: All functions above are assumed to be 0 for $t<0$.

i.e. when we say $\mathcal{L}\{e^{-at}\}$ we really mean $\mathcal{L}\{H(t)\,e^{-at}\}$

Ex: Solving an ODE.



"cruise control", v = velocity.

$$\boxed{m\dot{v} + bv = f}$$

Take Laplace transform of both sides:

$$\mathcal{L}\{m\dot{v} + bv\} = \mathcal{L}\{f\}$$

(write $\mathcal{L}\{v\} = V(s)$ and $\mathcal{L}\{f\} = F(s)$)

$$\Rightarrow \quad m\mathcal{L}\{\dot{v}\} + b\mathcal{L}\{v\} = \mathcal{L}\{f\}$$

$$\Rightarrow \quad m\big(sV(s) - v(0)\big) + bV(s) = F(s)$$

$$\Rightarrow \quad (ms+b)V(s) = F(s) + mv(0)$$

$$\Rightarrow \quad V(s) = \frac{1}{ms+b}F(s) + \frac{mv(0)}{ms+b}$$

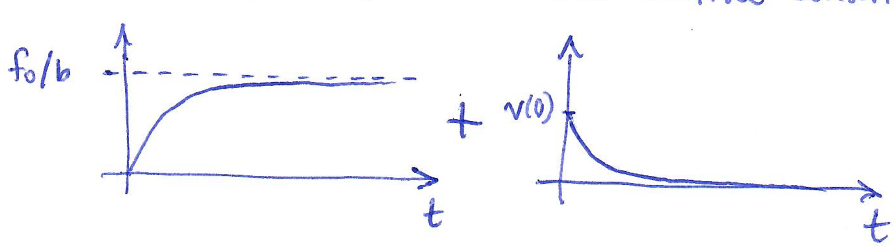
{ Suppose $f(t) = H(t)f_0$ (step function for force). Then $F(s) = \frac{f_0}{s}$ }

$$\Rightarrow \quad V(s) = \frac{f_0}{m}\cdot\frac{1}{s\left(s+\frac{b}{m}\right)} + \frac{v(0)}{s+\frac{b}{m}}$$

$\mathcal{L}^{-1}$

{ Now use the table on prev page to <u>invert</u> $\mathcal{L}$ and recover func. of t. }

$$\Rightarrow \quad v(t) = \underbrace{\frac{f_0}{m}\cdot\frac{m}{b}\left(1 - e^{-bt/m}\right)}_{\text{particular solution, depends on input } f} + \underbrace{v(0)\,e^{-(b/m)t}}_{\text{homogeneous solution, depends on initial condition } v(0)}$$

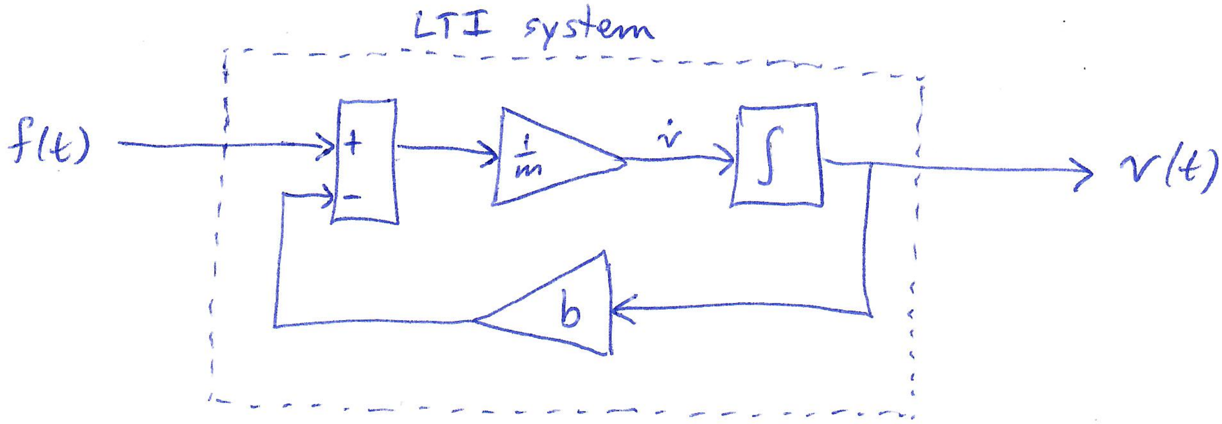


Let's not consider initial conditions for now. The homogeneous solution can always be found separately and added to the particular solution (a.k.a. forced response).

For the previous example (with $v(0) = 0$):

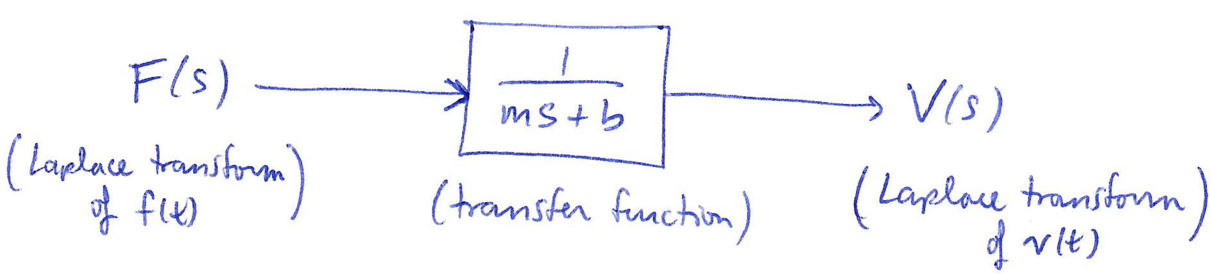
$$V(s) = \underbrace{\left(\frac{1}{ms+b}\right)}_{\text{this is called the transfer function}} F(s)$$

time domain:



Laplace domain:

⇕ equivalent!



In the Laplace domain (or "s-domain"), the blocks correspond to multiplication!

In general, we'll often use the letter "G" to denote a transfer function. For our system example, $G(s) = \frac{1}{ms+b}$.

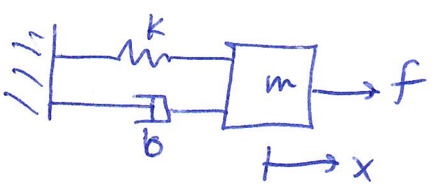
We can define the transfer function in general as:

$$G(s) = \frac{V(s)}{F(s)} = \frac{\mathcal{L}\{\text{output of system}\}}{\mathcal{L}\{\text{input of system}\}}$$

<u>Reminder</u> ☆ transfer functions are only defined for LTI systems! the linear properties are absolutely critical; it does not make sense to talk about the TF of a nonlinear system.

☆ the transfer function always represents input-output relationships when all initial conditions are zero.

<u>Ex</u>: spring-mass-damper.



$$m\ddot{x} + b\dot{x} + kx = f$$

$\mathcal{L}$ — set all initial conditions to zero. Then we have

$\dot{x} \to sX$

$\ddot{x} \to s^2X$

$\vdots$

$x^{(k)} \to s^kX$

↑ k^{th} derivative.

$$ms^2X(s) + bsX(s) + kX(s) = F(s)$$

$$\Rightarrow (ms^2 + bs + k)X(s) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \boxed{\frac{1}{ms^2 + bs + k}}$$

transfer function